8.2.1 Semi-empirical binding energy or mass formula

Since nuclear masses are accurately known experimentally, the nuclear binding energy $B.E$ is also known accurately. By using a semi-empirical approach, that is, an approach *based on experimental results*, Weizsäcker in 1935 proposed the following semi-empirical formula to achieve a quantitative and basic understanding of the nuclear binding energy, $B.E$ (in MeV) for the nucleus (Z, A).

$$B.E = a_v A - a_s A^{\frac{2}{3}} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_n \frac{(A-2Z)^2}{A} \pm \frac{\delta}{A^{3/4}} \quad (8.2.1)$$

with the constants or coefficients having typically the values all in MeV : $a_v = 14.0$, $a_s = 13.0$, $a_c = 0.60$, $a_n = 19$, and $\delta = 33.5$ for even-even or odd-odd nuclei and $\delta = 0$ for even-odd nuclei.

The mass formula has many applications, e.g., prediction of stability against β-decay for members of an isobaric family, explanation of fission by Bohr and Wheeler and calculation of stability limit against spontaneous fission etc.

We shall now describe the steps leading to the ***mass formula***. The liquid drop analogy of a nucleus, suggests that like the volume energy and surface energy of a liquid drop, there will be various contributions to the nuclear binding energy.

1. Volume energy term — The first term, $B_v = a_v A$, is the *volume effect* representing the volume energy of all nucleons. The larger the total number of nucleons A, the more difficult it is to remove an individual nucleon from the nucleus. Since the nuclear density is nearly constant, the nuclear mass is proportional to the nuclear volume, which again is proportional, for spherical nucleus, to R^3. But $R \propto A^{1/3} \Rightarrow R^3 \propto A$. So the volume energy, $B_v \propto A$. Thus the main contribution to $B.E$ comes from the total number of nucleons A and, as a first approximation,

$$\boxed{B_v = a_v A}$$

where a_v is a constant, called the *volume coefficient.*

• **From liquid drop analogy** — The energy needed for a complete evaporation of a liquid drop is the product of latent heat L and the mass of the drop M and is used to overcome all the molecular bonds, i.e., it equals the binding energy B of the drop.

$\therefore$ For a liquid drop,

$$B = LM = LmA$$

where m = mass of a molecule, A = number of molecules in the drop.

$$\therefore \; B/A = \text{constant} \Rightarrow B/A \text{ is } \textit{independent} \text{ of } A,$$

the total number of molecules in the drop—an *important feature* of any system (liquid drop or nucleus), where the range of interaction among the constituents is much less than the dimension of the system. By analogy with liquid drop, therefore, we expect for a nucleus a volume energy term.

Since, B/A = constant, the volume energy term is given by

$$B_v = \text{constant} \times A = a_v A$$

In discussing B_v, we assumed that the size of the liquid drop was so large that all molecules are almost fully surrounded by neighbours and hence is more strongly bound. This however is *not correct* for the surface molecules, which has fewer neighbours. The situation becomes particularly more important for light nuclei and is represented in Fig. 8.3 where a medium nucleus and a light nucleus are shown. While in the medium nucleus the ratio of surface nucleons and the total number of nucleons $\simeq 0.63$ $(= 12/19)$, it is as high as 0.88 $(= 6/7)$ in light nuclei. So we have a second term in the mass formula—the *surface energy term* that follows.

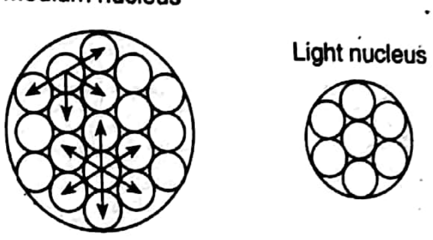

Fig. 8.3 Differences in surface energy of medium and light nuclei

2. Surface energy term — The second term, $B_s = a_s A^{2/3}$, is the *surface effect*, similar to the surface tension in liquids; like the molecules on the surface of a liquid, the nucleons at the surface of the nucleus are not completely surrounded by other nucleons. The total binding energy is thus reduced due to nucleons on the surface. This correction due to surface energy, B_s, is proportional to the surface area of the nucleus i.e. to $4\pi R^2$, for spherical nucleus of radius R. But $R \propto A^{1/3}$. So $B_s \propto A^{2/3}$.

$$\therefore \quad \boxed{B_s = a_s A^{2/3}}$$

where the constant. a_s is called the *surface coefficient.*

3. Coulomb energy term — The third term, B_c is the *Coulomb electrostatic repulsion* between the charged particles, protons, in the nucleus. Since each charged particle repulses all other charged particles, this term would be proportional to the possible number of combinations for a given proton number Z, which is $Z(Z-1)/2$. The energy of interaction between the protons is again inversely proportional to the distance of separation R. So the energy associated with Coulomb repulsion is

$$B_c = k\frac{Z(Z-1)}{R} = k\frac{Z(Z-1)}{r_0 A^{1/3}}$$

$$\text{or} \quad \boxed{B_c = a_c\frac{Z(Z-1)}{A^{1/3}}}$$

where R is replaced by $r_0 A^{1/3}$ and since this repulsive effect also dilutes the binding energy, it appears as a negative quantity in the semi-empirical mass formula.

4. Asymmetry energy term — The fourth term B_a, originates from the asymmetry between the number of protons and neutrons in the nucleus. For stable lighter nuclei, the number of protons is almost equal to that of neutrons : $N = Z$. As A increases, the symmetry of proton and neutron number is lost and the number of neutrons exceeds that of protons to maintain nuclear stability. This *neutron excess* i.e. excess of neutrons over protons, that is $N - Z$, is the measure of the *asymmetry* and it decreases the stability or $B.E$ of the medium or heavy nuclei.

The asymmetry energy, B_a, is directly proportional to (i) the *neutron excess*, $N - Z$ or $A - 2Z$ ($\because$ $A = N + Z$), present in asymmetric nuclei and (ii) the *fraction of nuclear volume* in which the excess neutrons are present. As the nuclear volume is proportional to A, the fractional volume of the nucleus in which excess neutrons are present will be proportional to $(N - Z)/A$ i.e., neutron excess per nucleon.

$$\therefore \quad B_a \propto (N - Z), \text{ and also}$$
$$\propto (N - Z)/A$$

$$\therefore \quad B_a = a_n \frac{(N - Z)^2}{A}$$

$$\boxed{B_a = a_n \frac{(A - 2Z)^2}{A}}$$

where a_n is a constant, called the *asymmetry coefficient*.

- Unlike in a liquid drop, there is in the nucleus the aspect of quantization of energy states of individual nucleons and the application of Pauli's principle.

If Z protons and N neutrons are put into a nucleus, the lowest Z-energy levels get filled up first. The *excess neutron* $(N - Z)$, by Pauli's principle, must go to higher unoccupied quantum states, as the first Z quantum states are already filled up with

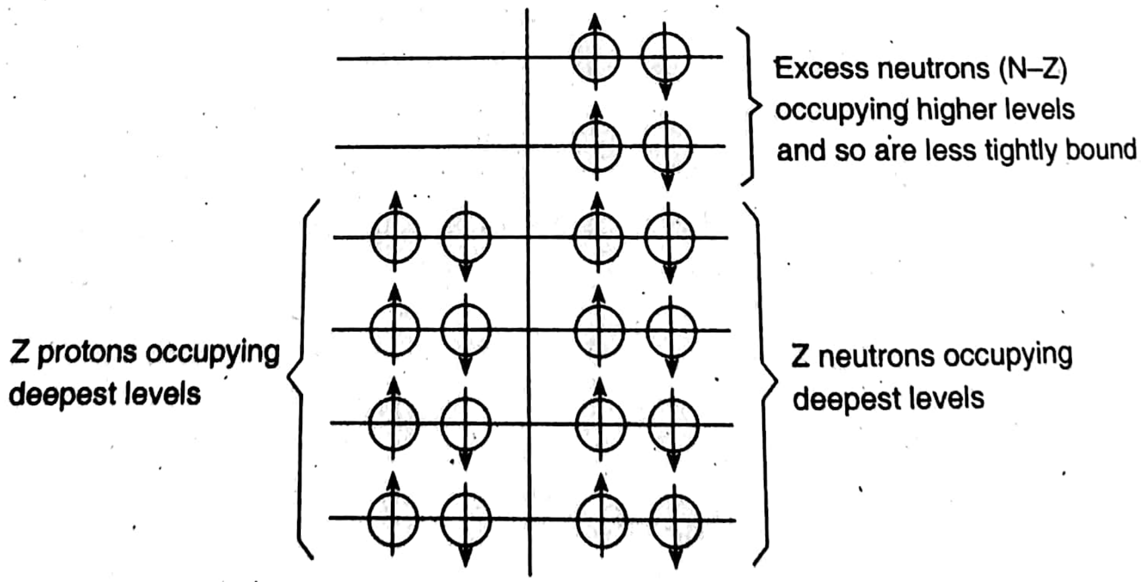

Fig. 8.4 Application of Pauli's principle to nucleon energy quantum states

protons and neutrons (Fig. 8.4). Consequently, the excess neutrons are less tighly bound than the first $2 \times Z$ nucleons, occupying the deepest energy levels. The asymmetry thus gives rise to a disruptive term B_a in nuclear binding energy — the *asymmetry term*.

By the incorporation of this purely quantum mechanical aspect in binding energy, the mass formula *goes beyond* the liquid drop analogy.

5. Pairing energy term — All the energy terms introduced so far involve a somewhat smooth variation of $B.E.$ with change in proton number Z or neutron number N. But $B.E./A$ vs. A plot shows a number of kinks and evidence of favoured pairings. For instance, nuclides with Z (or N) $= 2, 4, 8, 20, 50, 82$ and 126 (magic numbers) have larger $B.E.$-value. This fact is not taken into acount in a liquid drop model; intrinsic nucleonic spin and shell effects are disregarded. This omission demands a correction which is made in part by introducing the last term which is a pure corrective term, called the *pairing energy term*, B_p.

Nuclear data indicate that nuclei with ***even Z and even N are most stable***, whereas nuclei having ***odd Z and odd N. are least stable***, and nuclei with odd N and even Z, or even N and odd Z lie in between. Each of the protons and neutrons having spin $\frac{1}{2}$ form pairs with parallel and anti-parallel spins in even N- even Z type nuclei giving them a stable configuration. But in odd Z- odd N type nuclei, one unpaired proton and one unpaired neutron are left to make the nuclei less stable. So the pairing of spins increases the $B.E$ of even Z- even N type nuclei and decreases it in odd Z- odd N nuclei. Thus, the correction term B_p of pairing energy which is proportional to $A^{-3/4}$ is given by

$$\boxed{B_p = \frac{\delta}{A^{3/4}}}$$

where δ is a constant. This relation was determined empirically by Fermi.

No correction term however is necessary if A is *odd*, i.e. for A odd, $\delta = 0$. The constant δ is selected according to the following table.

Table 8.1 : *Classification of Stable Nuclides*

Z	N	A	No. of stable nuclei	δ	B_p
even	even	even	165	-33.5	$-\delta/A^{3/4}$
even	odd	odd	55	0	0
odd	even	odd	50	0	0
odd	odd	even	4	$+33.5$	$+\delta/A^{3/4}$

The binding energy $B.E$ of a nucleus is thus finally given by

$$B.E = a_v A - a_s A^{2/3} - \frac{a_c Z(Z-1)}{A^{1/3}} - a_n \frac{(A-2Z)^2}{A} \pm \frac{\delta}{A^{3/4}}$$

$$\therefore \quad f_B = \frac{B.E}{A} = a_v - \frac{a_s}{A^{1/3}} - \frac{a_c Z(Z-1)}{A^{4/3}} - a_n \frac{(A-2Z)^2}{A^2} \pm \frac{\delta}{A^{7/4}} \qquad (8.2.2)$$

where f_B is the *binding fraction*, i.e., the binding energy per nucleon.

The formula for the mass of the nucleus is given by

$$
\begin{aligned}
{}^{A}_{Z}M &= ZM_p + (A-Z)M_n - B.E/c^2 \\
&= ZM_p + (A-Z)M_n \\
&\quad -\frac{1}{c^2}\left[a_v A - a_s A^{\frac{2}{3}} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_n \frac{(A-2Z)^2}{A} + \frac{\delta}{A^{3/4}}\right] \qquad (8.2.3)
\end{aligned}
$$

The above formula (8.2.3) is known as the **semi-empirical mass formula** of Weizsächer.

Discussion — The five empirical constants or coefficients are evaluated using information about the $B.E.$ of nuclei which again is obtained from the accurate nuclear masses. Once the five constants are evaluated from five nuclear masses, we can use them to predict hundreds of other masses and reactions. Thus, using some empirical data, much of the nuclear behaviour becomes predictable. Hence (8.2.3) is known as *semi-empirical mass formula*.

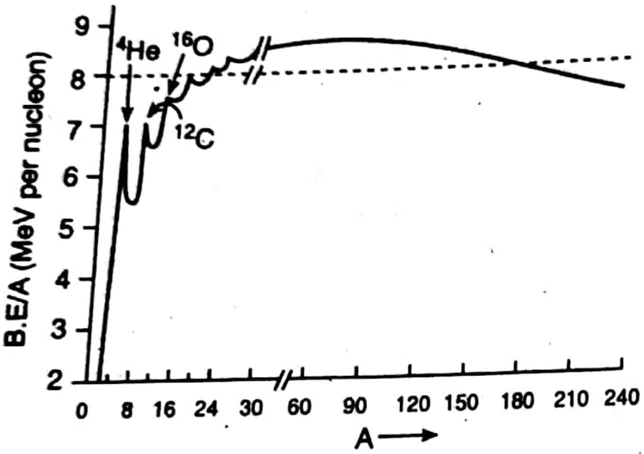

Fig. 8.5 Binding fraction curve

The $B.E/A$'s appear in Fig. 8.5 which is a plot of $B.E/A$ in MeV against the mass number A. With the exception of few irregularities such as ^{4}He, ^{12}C, ^{16}O etc. the curve is relatively smooth, rising sharply for small values of A. For values of $A \geq 30$, the binding energy is close to **8 MeV per nucleon**.

The relative contributions of the various effects in Weizsäcker's formula are shown schematically in Fig. 8.6.

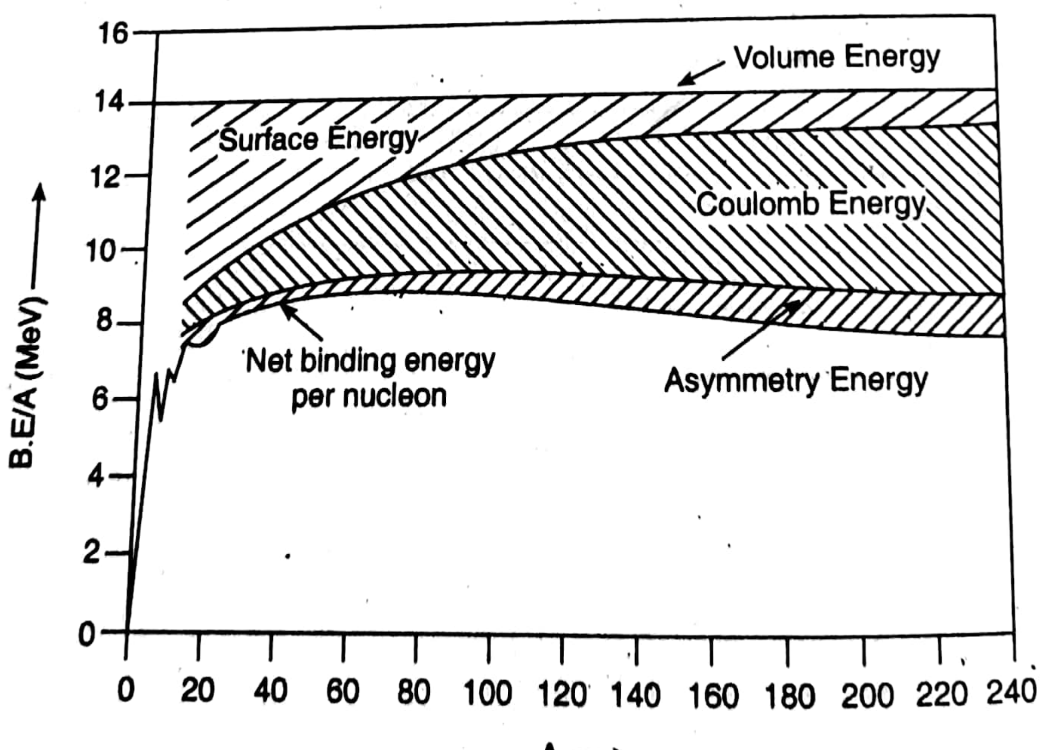

Fig. 8.6 Relative contributions of various effects in Weizsacker's formula

Note that (8.2.3) is quadratic in Z and thus for each A-value, there is a particular Z-value for which M is minimum. A family of different nuclides with same A-value is called an ***isobaric family***. The Z-value corresponding to minimum M is the *most stable member* of the isobaric family.

• Also note from Table 8.1 that while even-even nuclei are most stable and hence most abundant, the odd-odd nuclei are the least stable ones. Naturally, the odd-even nuclei are intermediate in respect of stability.

8.2.2 Applications of semi-empirical mass-formula

We give below a number of applications of the semi-empirical mass formula

1. Mass parabola : prediction of stability of nuclei against β-decay.

If $M(A,Z)$ be the atomic mass of an isotope of an element of atomic number Z and mass number A, then

$$M(A,Z) = ZM_p + NM_n - B.E. \tag{8.2.4}$$

where M_p, M_n are the masses of a proton and a neutron respectively.

Using (8.2.1), for $B.E$, the above equation becomes

$$M(A,Z) = ZM_p + (A-Z)M_n - a_v A + a_s A^{\frac{2}{3}} + a_c Z^2 A^{\frac{1}{3}} + a_n \frac{(A-2Z)^2}{A}, \tag{8.2.5}$$

neglecting δ and using Z^2 instead of $Z(Z-1)$ which appeared recently to be a better representation.

Introducing,

$$F_A = A(M_n - a_v + a_n) + a_s A^{\frac{2}{3}},$$

$$p = -4a_n - (M_n - M_p),$$

and

$$q = \frac{1}{A}(a_c A^{\frac{2}{3}} + 4a_n),$$

we obtain from equation (8.2.5) above :

$$M(A,Z) = F_A + pZ + qZ^2 \tag{8.2.6}$$

which is an *equation to a parabola* for a given A (i.e., for a given isobaric line) and is known as the **mass parabola** (Fig. 8.7).

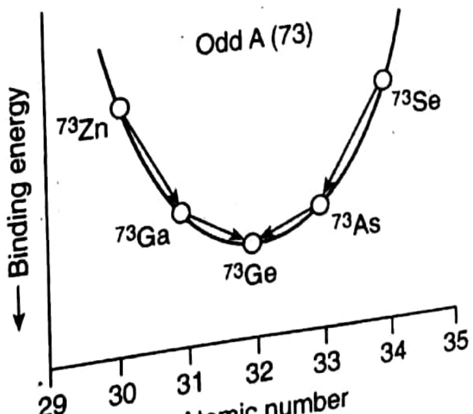

Fig. 8.7 Mass parabola

The lowest point of the parabola, $Z = Z_A$, is obtained by differentiating $M(A,Z)$ with respect to Z for a given A, and equating the same to zero.

$$\therefore \left(\frac{\partial M}{\partial Z}\right)_A = p + 2qZ = 0 \text{ at } Z = Z_A, \text{ whence,}$$

$$Z_A = -\frac{p}{2q} = \frac{(M_n - M_p + 4a_n)A}{2(a_c A^{\frac{2}{3}} + 4a_n)} \tag{8.2.7}$$

$$\therefore \text{ From (8.2.6)} \quad M(A, Z_A) = (F_A + pZ_A + qZ_A^2)$$

$$= F_A + p\left(-\frac{p}{2q}\right) + q\left(\frac{p^2}{4q^2}\right), \text{ using (8.2.7)}.$$

$$= F_A - \frac{p^2}{4q}$$

$$\therefore \quad M(A, Z) - M(A, Z_A) = (F_A + pZ + qZ^2) - \left(F_A - p^2/4q\right)$$

$$= p^2/4q + pZ + qZ^2$$

$$= q\left(\frac{p^2}{4q^2} + \frac{p}{q}Z + Z^2\right)$$

$$= q\left(Z + \frac{p}{2q}\right)^2$$

$$= q(Z - Z_A)^2 = \text{a positive quantity.}$$

That is, the mass parabola for a given isobar ($A =$ constant) has the lowest point at $Z = Z_A$. Since $M(A, Z_A)$ has the smallest value for a given A, this nucleus has the *largest B.E. and is the most stable among the isobars* for the given A.

On substituting the values of M_n, M_p, a_c and a_n in (8.2.7),

$$Z_A = A/(1.98 + 0.015A^{2/3})$$

which does not usually give an integral value for Z_A. In most cases, the value of Z nearest to Z_A gives the *actual* stablest nucleus for a given A. ***All isobars having B.E. less than the most stable one will lie on the two arms of the parabola.*** Their masses will be greater than that of the stable isobar and they will decay by emission of β^-, β^+ or K-capture. The isobars to the left of the stable one, decay by β^- emission as they have fewer protons than the stable one, while those to the right having an excess proton decay by β^+ emission or K-capture, or by both.

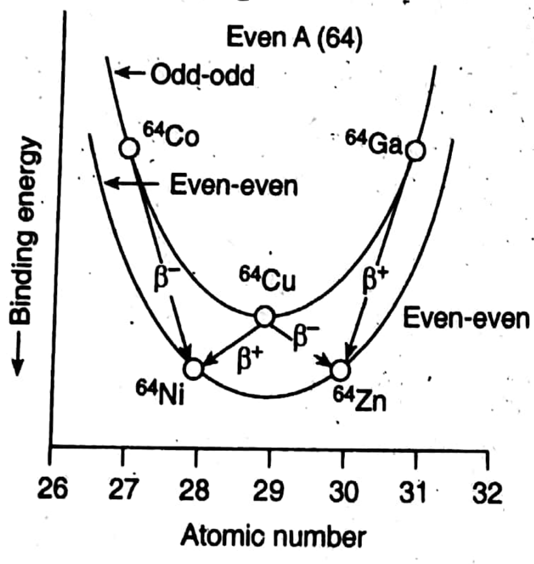

Fig. 8.8 Mass parabolas

$$^{73}_{30}\text{Zn} \xrightarrow{\beta^-} {}^{73}_{31}\text{Ga} \xrightarrow{\beta^-} {}^{73}_{32}\text{Ge} \ \text{(stable)}$$

$$^{73}_{34}\text{Se} \xrightarrow{\beta^+, K} {}^{73}_{33}\text{As} \xrightarrow{K} {}^{73}_{32}\text{Ge} \ \text{(stable)}$$

- So long we did not take δ into account. If we do, the *mass parabolas* for different isobars fall into *two groups* depending on if A is odd or even. For *odd A*, a *single parabola* for each A is obtained since for odd A, $\delta = 0$. For *even A*, we get *two parabolas* for the same A: one is for even Z (even-even nucleus), and the other for odd Z (odd-odd nucleus). Since δ is subtracted for odd-odd nuclei and added for even-even, the parabola for odd-odd nuclei is above that for even-even (Fig. 8.8). The odd Z- even A nuclides